Finite element ocean modeling on unstructured prismatic meshes*

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1 Introduction

Numerical ocean models are currently used in two very distinct applications. High-resolution ocean models (around 10 km) are now able to resolve most of the energetic mesoscale variability. They give a consistent description of the ocean circulation down to the first baroclinic deformation radius. However, given the huge computing requirement, these models are restricted to relatively short integration times (at most a few decades) or single ocean basins and are therefore unsuitable for climate studies. For the latter, ocean climate models, with a much lower horizontal grid resolution (around 100 km), afford much longer integration times. These models are often run as part of global atmosphere-ocean climate models, coupled with biogeochemical cycles. However, due to the poor resolution, these models are seriously misrepresenting many important oceanic processes. A convergence of both classes of models remains a very distant goal if we rely on the increasing computing power alone. Bridging this gap requires a revolutionary change in the algorithmic nature of ocean models. The use of unstructured meshes is believed to be the catalyst to that revolution. These meshes form the basis of so-called second-generation ocean models.

2 Short historical perspective

The finite element method lends itself to the use of unstructured meshes and we might wonder why almost three decades have elapsed since the work by Fix (1975) before intensive finite element ocean model developments started. The finite element method has always been the option of choice for elliptic problems, while having more troubles solving hyperbolic problems such as advection-dominated flows and wave propagation problems. In particular, the simple two-dimensional shallow-water equations are challenging in that a naive discretization of velocity and elevation with linear triangles – a staggering akin to the A-grid – presents spurious pressure modes and is thus numerically unstable. From this point on, two research tracks have been followed.

Rather than finding a mixed finite element pair that did not support spurious modes, the wave continuity equation method consists in manipulating the primitive shallow-water equations to form a wave equation for the elevation (Lynch and Gray, 1979). This approach has the advantage of circumventing the numerical instabilities associated with using the same interpolation for both variables. It does, however, have two caveats. First, the primitive form of the elevation equation is sacrificed to form a wave equation. Hence the discrete form of this equation is no longer satisfied and consistency between this equation and the three-dimensional continuity equation breaks down which may eventually imply tracer conservation breakdown (Dawson et al., 2006).

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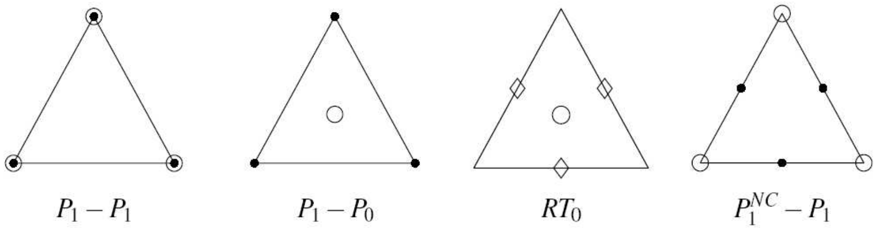

Figure 1: Two-dimensional variable staggering on triangular elements (○: elevation, •: full velocity, ♢: normal velocity). The $P_1 - P_1$, $P_1 - P_0$ and RT_0 pairs are essentially equivalent to the A, B and C grids, respectively. The $P_1^{NC} - P_1$ pair has similarities with the CD grid but has more degrees of freedom for the elevation field. The first two pairs are not usable for finite element ocean modeling while the last two have the best numerical properties.

Second, the method suffers from advective instabilities (Kolar et al., 1994). These problems are major obstacles that prevent the method to be applied to large-scale ocean problems.

In parallel to these studies, a lot of effort has been directed towards finding a mixed finite element pair for the primitive shallow-water equations that does not support spurious oscillations, which culminated with the papers by Le Roux et al. (1998) and Hanert et al. (2003). Aware of the limitations of the wave continuity method and urged to develop primitive equations finite element ocean models, research towards this goal intensified (Le Roux, 2005; Le Roux et al., 2005; Walters, 2006; White et al., 2006; Le Roux et al., 2007). Early issues of the method often cited as reasons not to use it – such as spurious oscillations, unphysical wave scattering due to the unstructured character of the mesh and lack of mass conservation – lose momentum and concrete developments are more than ever well on track.

3 Variables staggering

We focus on prismatic meshes, built by vertically extruding two-dimensional triangular unstructured meshes. The mesh anisotropy is hard-coded and essentially translates the fact that large-scale ocean flows present, to a large extent, the same anisotropy. We first restrict ourselves to studying the variables staggering in two dimensions separately from the vertical dimension and then extend these considerations to the third dimension.

3.1 Two-dimensional variables staggering

The material presented here is a very succinct summary of the much more detailed work by other authors to which the interested reader may refer (Hua and Thomasset, 1984; Le Roux et al., 1998; Le Roux, 2001; Hanert et al., 2003; Le Roux, 2005; Le Roux et al., 2005; Walters, 2006; White et al., 2006; Le Roux et al., 2007). Built on past experience on the effect of staggering on finite difference solutions to the shallow-water equations, the idea was to verify whether bad and good qualities carried over to finite element discretizations (Figure 1). The RT_0 and $P_1^{NC} - P_1$ pairs are the most promising for finite element ocean modeling, with no spurious modes (except at low resolution for RT_0, which can easily be filtered out with momentum diffusion) and good numerical dispersion relationships.

3.2 Three-dimensional variables staggering

The three-dimensional spatial staggering is constrained by the following requirements:

1. the three-dimensional computational domain must be mobile in the vertical to adapt to the free-surface motion and to correctly handle freshwater fluxes
2. the volume of the (Boussinesq) ocean must be conserved,

3. any tracer must be globally conserved (global conservation),

4. the tracer equation must preserve constants (local consistency).

These constraints allow us to select the proper elements for the vertical velocity (w) and the tracers (C), which yield the following sufficient conditions to satisfy all requirements (White et al., 2007).

1. The same element must be used for w and C.

2. The nodes location in the horizontal must be the same for the elevation (η) and w.

3. The two previous statements also imply that the nodes location in the horizontal must be the same for η, w and C.

4. In the vertical, the nodes location for w and C is unconstrained, yet it must be identical for both variables.

Therefore, given that a stable discretization should be used in two dimensions, the associated choice strongly constrains which elements can be used in three dimensions.

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