CABARET in the ocean gyres: a novel high-resolution computational method for fluid mechanics

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Abstract:

In many aspects mesoscale oceanic eddies, operating on the lengthscales of O(1-100) km are analogous to the cyclones and anticyclones that constitute the atmospheric weather phenomenon. The problem of resolving these eddies in a dynamically consistent way is very important for ocean modelling and, therefore, for global climate predictions. For achieving high Reynolds number (Re) simulations, which are one of the goals of the ocean modelling, the models have to resolve all important scales of motion.

Modern ocean models enter a new phase in which eddies will be, at least, permitted in the numerical

simulation. For such models advection scheme is a very important component. A crucial element of numerical advection scheme is its ability to propagate finite-amplitude and -phase disturbances on a discrete grid either without generating spurious short-wave oscillations, because of not preserving the correct dispersion relation i.e., dispersion error, or any considerable damping of the amplitude i.e., dissipation error

A new high-resolution Eulerian numerical method is proposed for modelling quasigeostrophic ocean dynamics in eddying regimes. The method is based on a novel, second-order low-dissipative and low-dispersive conservative advection CABARET scheme.

1. Numerical method

Consider a scalar conservation law

(1)

on a finite-difference grid which is non-uniform in space $x_{i+1} - x_i = h_{i+1/2}$ and time $t^{n+1} - t^n = \tau^{n+1/2}$.

First, take a half time step using forward-time central approximation,

$$\frac{u_C - u_E}{\tau^{n+1/2}} + \frac{f_4 - f_5}{h_{i+1/2}} = 0. \qquad (2)$$

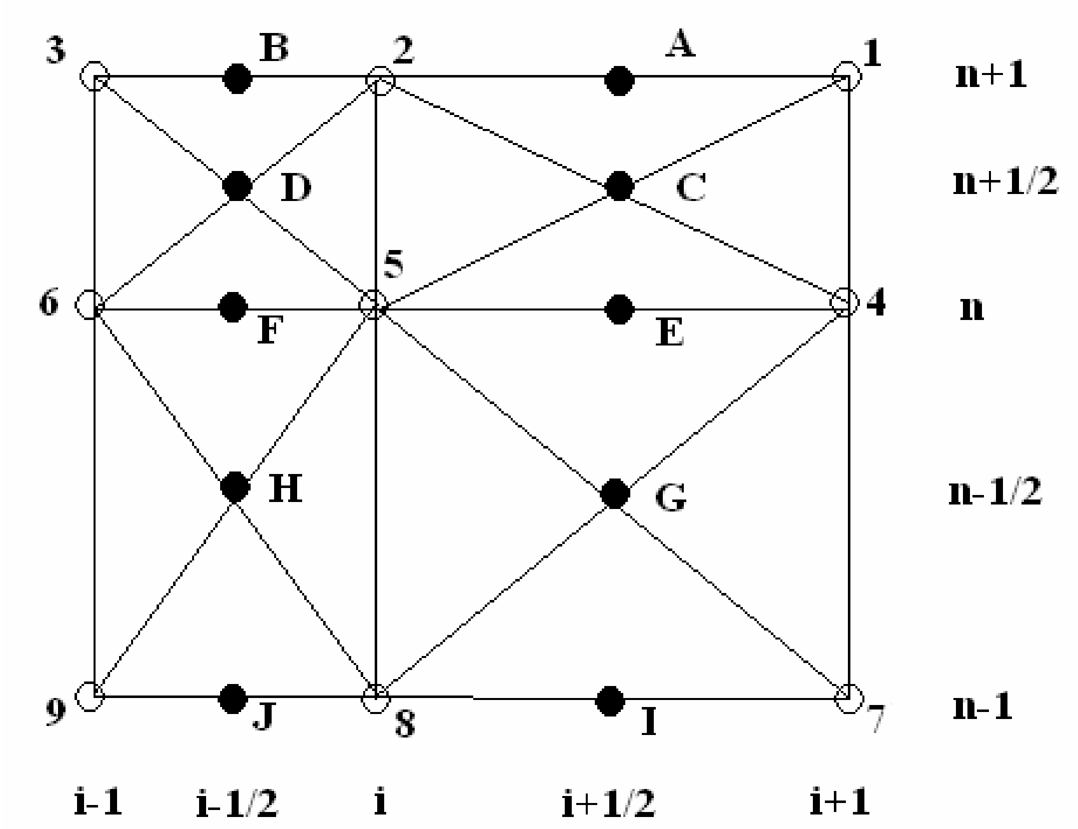
Then use the backward-time central-space approximation,

$$\frac{u_A - u_C}{\tau^{n+1/2}} + \frac{f_1 - f_2}{h_{i+1/2}} = 0. \qquad (3)$$

By symmetry, the scheme is second-order accurate for a sufficiently accurate evaluation of fluxes. Let’s assume that the sign of the wave speed, $\partial_u f(u)$, is positive everywhere. Then, we determine $f_1 = f(u_1)$ by assuming that

$$u_1 = 2u_C - u_5 . \qquad (4)$$

With this choice, the entire scheme (2)-(4) is time-reversible. It is also second order accurate, regardless of the gird non-uniformity in space and time, and it is non-dissipative.



For making the CABARET solution non-oscillatory, a simple tunable-parameter-free flux limiter is introduced through a non-linear correction procedure for the flux variables:

$$
\begin{aligned}
& u_1 = 2u_C - u_5; \\
& if\,(u_1 > Max(u_4, u_E, u_5)) \;\; u_1 = Max(u_4, u_E, u_5); \\
& if\,(u_1 < Min(u_4, u_E, u_5)) \;\; u_1 = Min(u_4, u_E, u_5).
\end{aligned} \qquad (4a)
$$

The nonlinear correction procedure is based on the maximum principle (e.g., Boris et al., 1975; Harten et al., 1987) for characteristic wave that arrives at point 1 from the solution domain dependency 4-E-5, and that is approximated using the 3-point stencil within one cell in space.

The CABARET approach is based on several important ideas:

(i) *Fully discrete/Lagrangian property*: approximation of the entire material derivative

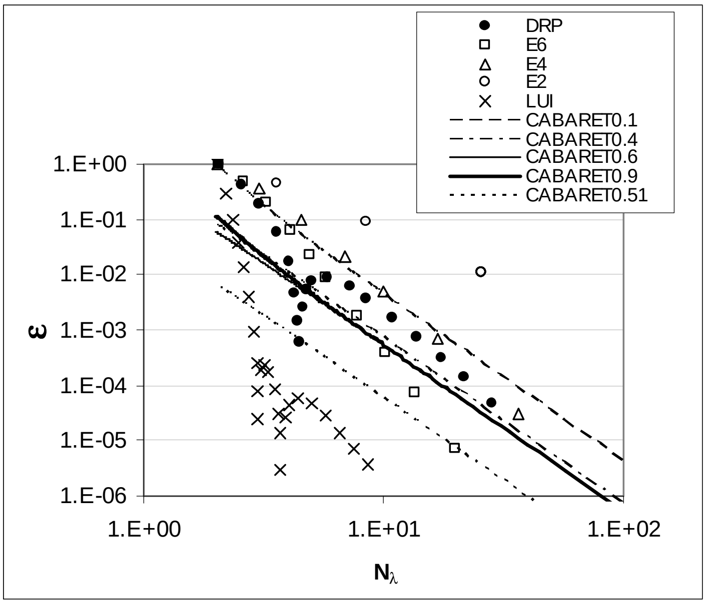
$$L = \left\langle \partial_t + \partial_u f(u) \partial_x \right\rangle,$$

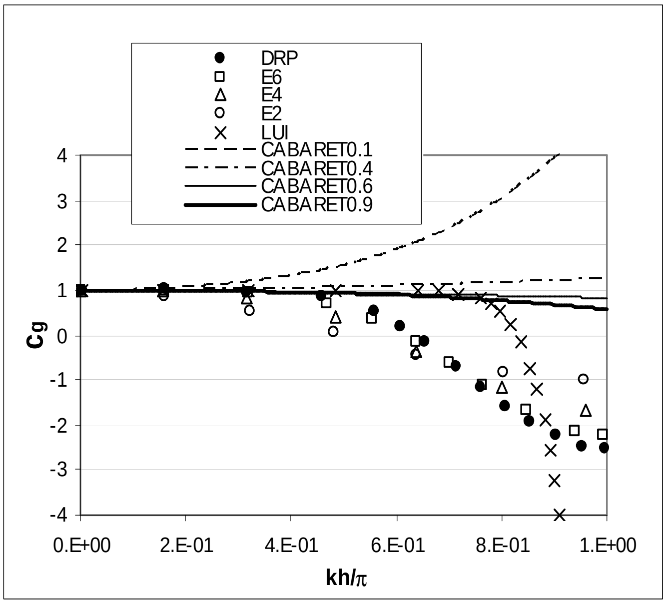
rather than optimisation of the time and space discretisations separately, as in the standard Eulerian schemes;

$$L_x = \left\langle \partial_u f(u) \cdot \partial_x \right\rangle, L = \left\langle \partial_t \right\rangle + L_x.$$

(ii) *Low dispersion and no-dissipation error* of the underlining finite-difference scheme;

Phase speed error **Group speed error**

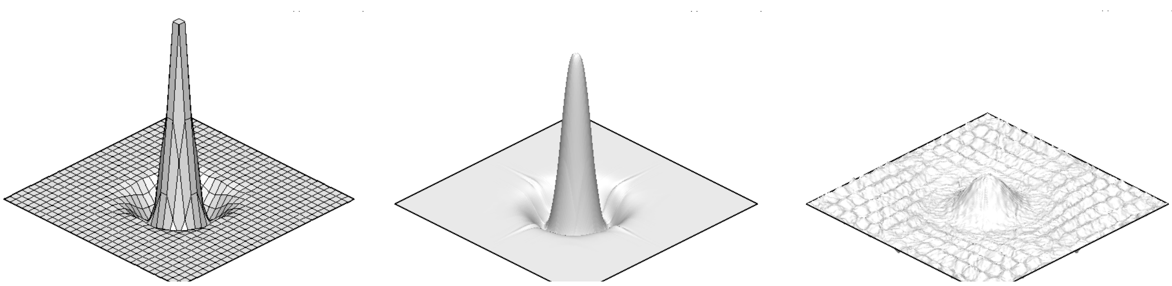




- Test: 2D compressible Euler equations

Steady zero-circulation vortex: vorticity plots

CABARET **3rd order MUSCL-TVD**



Coarse grid, long time: good

Fine grid, short time: clipping

Fine grid, long time: smearing

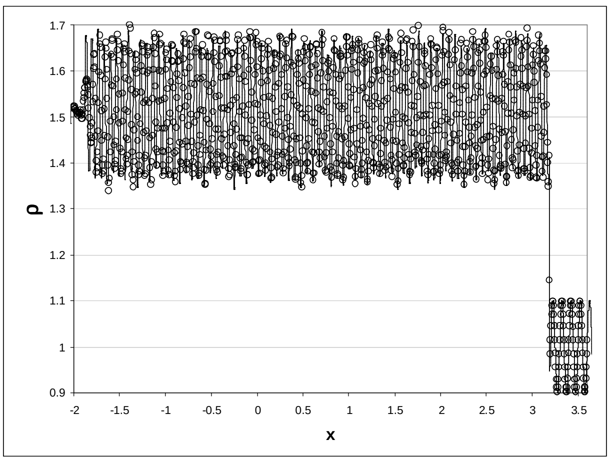
(iii) *Non-oscillatory property*: enforcing the maximum principle on the solution as the means for efficient treatment of the underresolved scales;

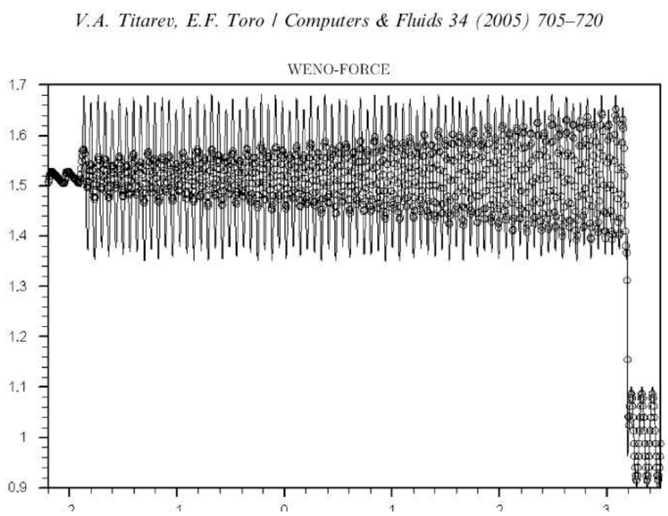
- Modified Shu-Osher shock/turbulence interaction problem

$$(\rho, u, p) = \begin{cases} (1.515695, 0.523346, 1.80500), & x < -4.5, \\ (1 + 0.1 \sin 20\pi x, 0.0, 1), & x > -4.5, \end{cases}$$

Density profile at t=5:

CABARET **4-th order WENO**





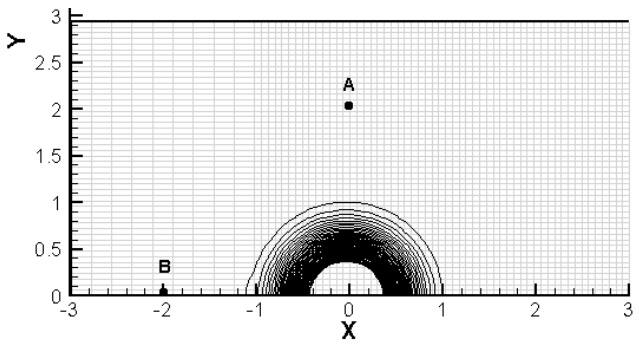
Good

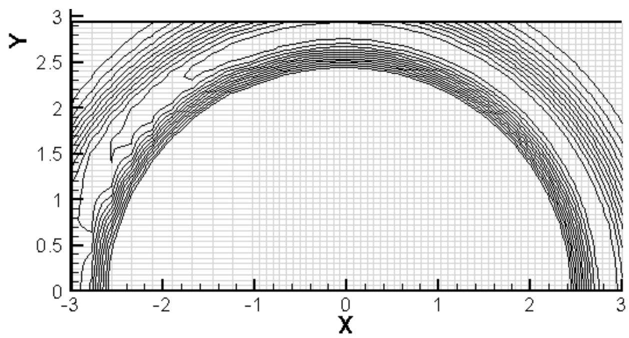
Attenuation of short scales

(iv) Ease of implementation for complicated boundary conditions and non-uniform grids due to the *compactness* of stencil in space and time (and also reduced CPU cost per time iteration)

- Test: 2D Euler equations

Acoustic Gaussian pulse propagation on a non-uniform grid





No spurious reflection from coarse grid/fine grid interface

2. Double-gyre problem

The QG model of the wind-driven double-gyre circulation is considered in a midlatitude closed basin, which is in the shape of a square with north-south and east-west rigid walls ([L x L] = 3840 x 3840 km). This model simulates not only the subpolar and subtropical ocean gyres but also the nonlinear western boundary currents, such as the Gulfstream or Kurosio, and their eastward jet extensions (e.g., Holland, 1978). The model stratification is represented by stacked isopycnal layers that are dynamically coupled through pressure fluctuations. The governing equations constitute the system of material conservation laws for potential vorticity (PV) anomaly , with a source term due to the meridional gradient of the Coriolis parameter and with the additional source terms due to the lateral viscosity, bottom friction, and the wind forcing.

The system of the quasilinear hyperbolic-type (e.g., Rozhdestvensky and Yanenko, 1978) equations for PV anomaly and the associated elliptic equations for velocity streamfunctions are:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\zeta_q + \frac{\partial}{\partial x}\left(u_q\zeta_q\right) + \frac{\partial}{\partial y}\left(v_q\zeta_q\right) = F_q, \quad q = \overline{1..3}, \\
&F_q = \delta_{1q}\cdot f_{wind} - \beta\cdot v_q + \delta_{3q}\cdot\mu_{bot}\nabla^2\psi_q + \mu_{eddy}\nabla^2 D_q, \\
&u_q = \frac{\partial\psi_q}{\partial y}, \quad v_q = -\frac{\partial\psi_q}{\partial x}, \\
&\zeta_1 = \nabla^2\psi_1 - s_1\left(\psi_1 - \psi_2\right), D_1 = \zeta_1 + s_1\left(\psi_1 - \psi_2\right), \\
&\zeta_2 = \nabla^2\psi_2 - s_{21}\left(\psi_2 - \psi_1\right) - s_{22}\left(\psi_2 - \psi_3\right), D_2 = \zeta_2 + s_{21}\left(\psi_2 - \psi_1\right) + s_{22}\left(\psi_2 - \psi_3\right), \\
&\zeta_3 = \nabla^2\psi_3 - s_3\left(\psi_3 - \psi_2\right), D_3 = \zeta_3 + s_3\left(\psi_3 - \psi_2\right), \\
&\beta, \mu_{bot}, \mu_{eddy}, s_1, s_{21}, s_{22}, s_3 = const > 0,
\end{aligned}
$$

where q=1,2, and 3 denote the top, intermediate, and the bottom isopycnal layers, respectively; δ_{kq} is the Kronecker symbol; β is the meridional gradient of the Coriolis parameter; $f_{wind} = f_{wind}(x, y)$ is the idealised steady L

$$f_{wind}(x,y)=\begin{cases} A\sin\left(\dfrac{\pi\cdot y/L}{y_0/L}\right) & \text{for } \ 0\le y<y_0, \\ -A\sin\left(\dfrac{\pi\cdot\left(y-y_0\right)/L}{1-y_0/L}\right) & \text{for } y_0\le y\le L, \end{cases}$$

$$y_0/L=0.5+0.2\cdot(x/L-0.5), \quad A=-2\pi\tau_0/(0.9\cdot L\cdot\rho_1);$$

$$0\le x/L\le 1, \quad 0\le y/L\le 1,$$

where the wind stress amplitude is $\tau_0 = 0.8N\cdot m^{-2}$, and the upper-ocean density is $\rho_1 = 10^3 kg/m^3$ (e.g., Berloff et al., 2007).

At the closed-basin boundary, Γ, the partial-slip condition is imposed:

$$\left(u_q n_x + v_q n_y\right)\Big|_{\Gamma} = 0, \ \left(\frac{\partial}{\partial n}\left(u_q n_x + v_q n_y\right) - \alpha^{-1}\cdot\left(u_q n_x + v_q n_y\right)\right)\Bigg|_{\Gamma} = 0,$$

$$\alpha=\text{const}>0, \ q=\overline{1..3}$$

where (n_x, n_y) are the Cartesian components of the normal unit vector. This boundary condition implies that the tangential velocity component at the wall corresponds to a prescribed exponential-decay law based on the characteristic boundary layer thickness, α. This condition corresponds to the mixed Dirichlet-Neumann boundary condition. In the ocean modelling practice, it is a conventional, though not fully justified, parameterisation for dynamically unresolved processes near the ocean coasts.

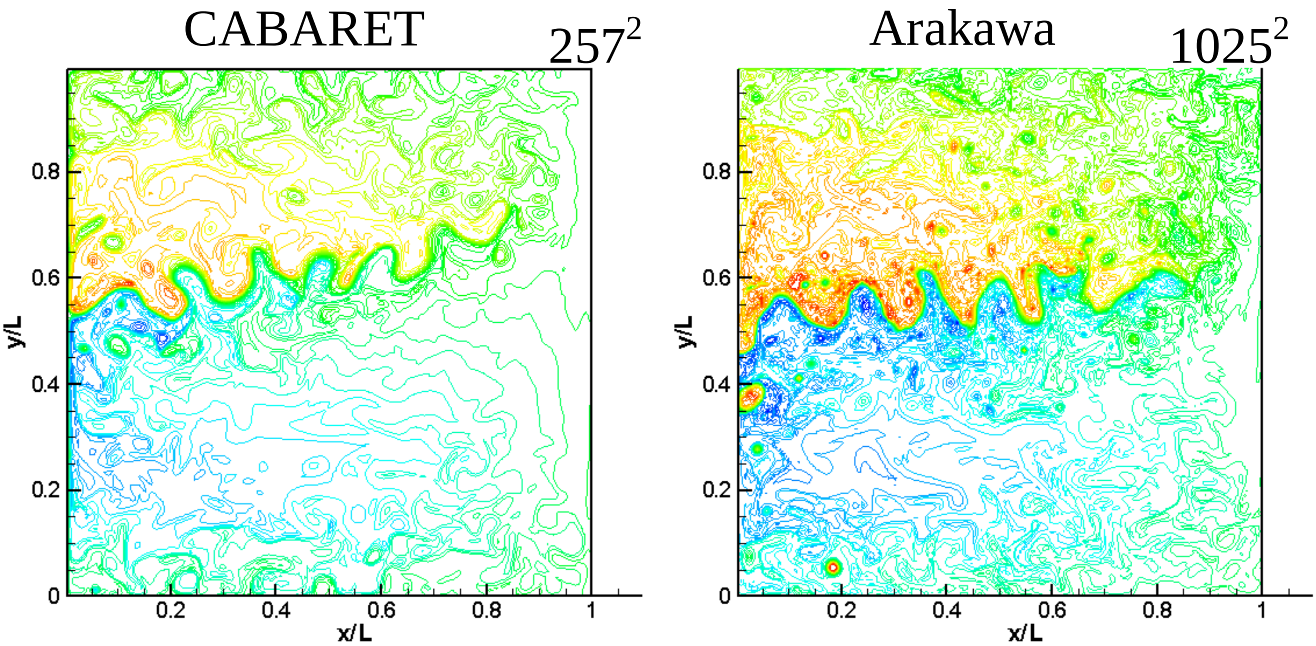
This is applied together with the integral mass conservation constraints (McWilliams, 1977),

$$\frac{\partial}{\partial t}\iint\left(\psi_1-\psi_2\right)\,dxdy=0 \text{ and } \frac{\partial}{\partial t}\iint\left(\psi_2-\psi_3\right)\,dxdy=0.$$

3. Results of QG Modelling: comparison with the conservative Arakawa method.

3 layer QG model, PV anomaly contour levels

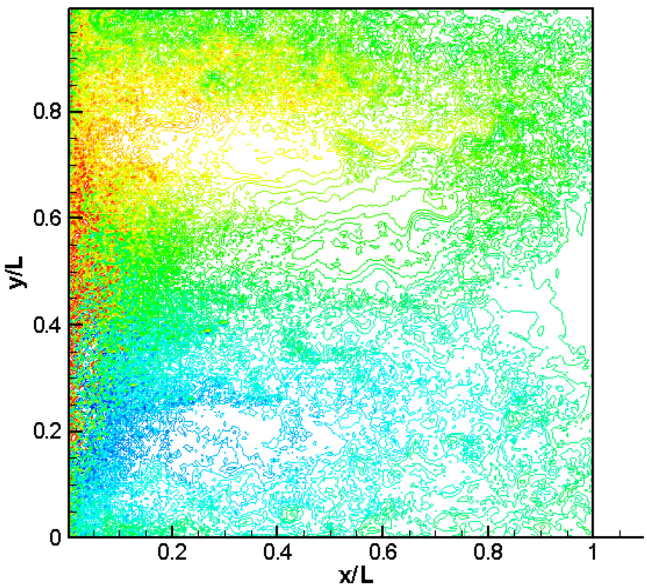
- Coarse-grid CABARET solution is similar to the fine-grid Arakawa solution, **Re=13300**



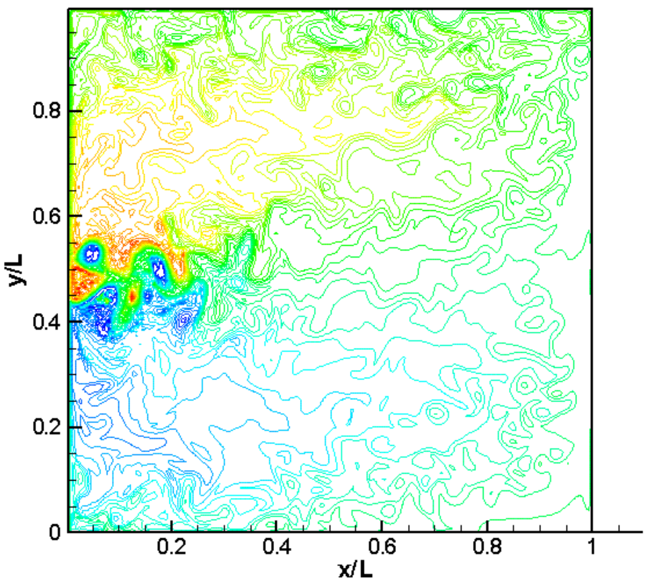
- Does the further parameterization help to improve the coarse-grid Arakawa solution?

- **Yes**, using of large eddy viscosity helps reduce the dispersion error and improves the EJ solution

Arakawa, Re=13300 257^2



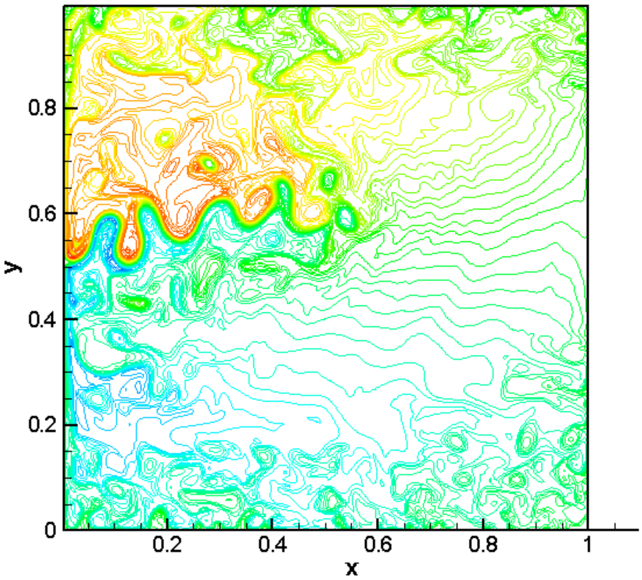
Arakawa, Re=1600 257^2



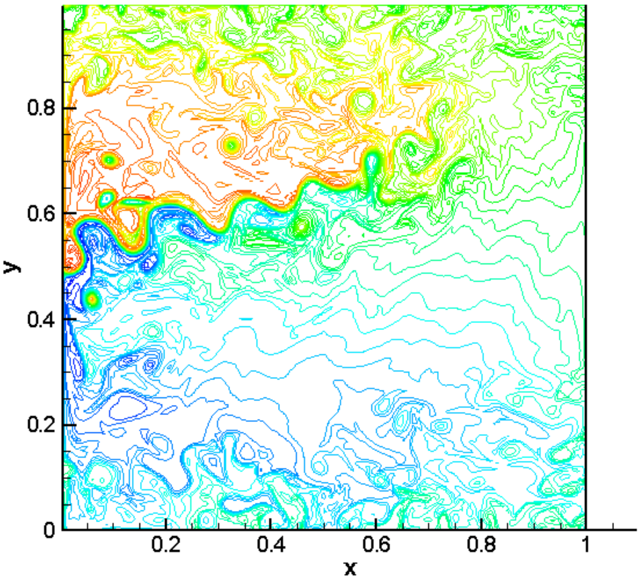
10 layer QG model, Re=1600, PV anomaly contour levels

- Coarse-grid CABARET solution is close to the converged solution

CABARET 257^2

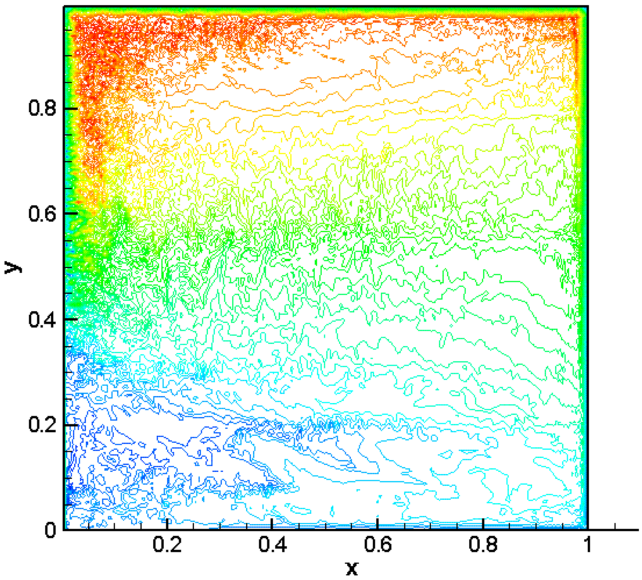


CABARET 513^2

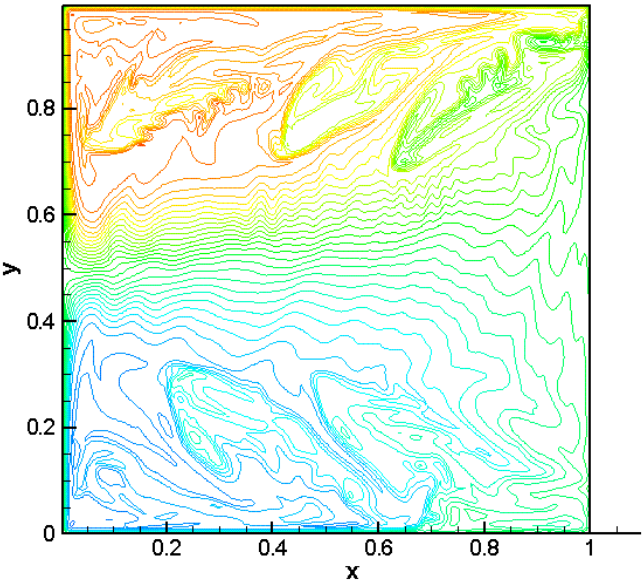


- Enhanced eddy viscosity does **NOT** fix the dispersion error problem with the coarse-grid Arakawa solution

Arakawa, Re=1600 257^2



Arakawa, Re=200 257^2



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- **Karabasov, S.A., Berloff, P.S., and Goloviznin, V.M. CABARET in the Ocean Gyres, Submitted to Journal of Ocean Modelling.**